SENSITIVITY OF GAUSSIAN BAYESIAN NETWORKS TO INACCURACIES IN THEIR PARAMETERS

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Def. A ***Bayesian network*** is a pair $(\mathcal{G},\mathcal{P})$ where $\mathcal{G}$ is a directed acyclic graph (DAG), where nodes represent random variables $\mathbf{X}=\{X_1,\ldots,X_n\}$ and edges represent probabilistic dependences, and $\mathcal{P}=\{p(x_1|pa(x_1)),\ldots,p(x_n|pa(x_n))\}$ is a set of conditional probability distributions (one for each variable) where $pa(x_i)$ is the set of parents of node X_i in $\mathcal{G}$.

The set $\mathcal{P}$ defines the associated joint probability density as

$$p(x)=\prod_{i=1}^{n} p(x_i \mid pa(x_i))$$

Def. A ***Gaussian Bayesian network*** is a Bayesian network over $\mathbf{X}=\{X_1,\ldots,X_n\}$ where the joint probability density is a multivariate normal distribution $N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})$, with $\boldsymbol{\mu}$ the n-dimensional mean vector and $\boldsymbol{\Sigma}$ the n x n positive definite covariance matrix, with the dependence structure of $\mathcal{G}$.

If there is any information about the state of a variable, the ***evidence propagation*** updates the probability distribution of the rest of variables with this information or evidence.

In Gaussian Bayesian networks the evidence propagation is performed by computing the conditional probability density of a multivariate normal distribution given the set of evidential variable $\mathbf{E}=\mathbf{e}$. Then, considering the partition $\mathbf{X}=(\mathbf{Y},\mathbf{E})'$ where $\mathbf{Y}$ is the set of non-evidential variables, the conditional probability distribution of $\mathbf{Y}$, given the evidence $\mathbf{E}=\mathbf{e}$, is a multivariate normal distribution, given by

$$\mathbf{Y}\mid\mathbf{E}=\mathbf{e}\approx N(\mathbf{y}\mid\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E}=\mathbf{e}},\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}=\mathbf{e}})$$ where

$$\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E}=\mathbf{e}}=\boldsymbol{\mu}_{\mathbf{Y}}+\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}(\mathbf{e}-\boldsymbol{\mu}_{\mathbf{E}})$$

$$\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}=\mathbf{e}}=\boldsymbol{\Sigma}_{\mathbf{YY}}-\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Sigma}_{\mathbf{EY}}$$

SENSITIVITY ANALYSIS

· This sensitivity analysis is based on the comparison between two different models:

- The *original model* $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})$
- The *perturbed model* $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu}^{\boldsymbol{\delta}},\boldsymbol{\Sigma}^{\boldsymbol{\Delta}})$, obtained after adding a *mean vector perturbations* $\boldsymbol{\delta}$ or a *covariance matrix perturbations* $\boldsymbol{\Delta}$ to the inaccurate elements of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, respectively.

· With these two models the evidence propagation is performed, obtaining the networks outputs as the conditional distributions of the set of variables of interest $\mathbf{Y}$ given $\mathbf{E}=\mathbf{e}$.

· Both networks outputs are compared by computing the **Kullback-Leibler divergence**.

KULLBACK-LEIBLER DIVERGENCE

To compare the networks outputs obtained for the original model and for the perturbed model, this expression for the Kullback-Leibler (*KL*) divergence is used

$$KL^{\mathbf{p}_j}(f,f^{\mathbf{p}_j})=E_f\left[\ln\frac{f}{f^{\mathbf{p}_j}}\right]=$$

$$=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{p}_j}\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{p}_j}\right)^{-1}\right)+\left(\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E},\mathbf{p}_j}-\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E}}\right)^T\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E},\mathbf{p}_j}\right)^{-1}\left(\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E},\mathbf{p}_j}-\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E}}\right)-\dim(\mathbf{Y})\right]$$

being f the conditional probability density obtained for the original model after the evidence propagation and $f^{\mathbf{p}_j}$ the conditional probability density obtained for the perturbed model, after the evidence propagation.

PERTURBED MODELS

Five different perturbed models are going to be compared with the original model. Those perturbed models are obtained by considering the partition of the mean vector perturbations $\boldsymbol{\delta}$ and the covariance matrix perturbations $\boldsymbol{\Delta}$ as

$$\boldsymbol{\delta}=\begin{pmatrix}\boldsymbol{\delta}_{\mathbf{Y}}\\ \boldsymbol{\delta}_{\mathbf{E}}\end{pmatrix}\quad\text{and}\quad\boldsymbol{\Delta}=\begin{pmatrix}\boldsymbol{\Delta}_{\mathbf{YY}} & \boldsymbol{\Delta}_{\mathbf{YE}}\\ \boldsymbol{\Delta}_{\mathbf{EY}} & \boldsymbol{\Delta}_{\mathbf{EE}}\end{pmatrix}$$

Therefore, there can be uncertainty about

-The means of the variables of interest, being the perturbed model $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu}^{\boldsymbol{\delta}\mathbf{Y}},\boldsymbol{\Sigma})$, or about the means of the evidential variables, with the perturbed model $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu}^{\boldsymbol{\delta}\mathbf{E}},\boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu}^{\boldsymbol{\delta}_{\mathbf{Y}}}=\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{Y}}+\boldsymbol{\delta}_{\mathbf{Y}}\\ \boldsymbol{\mu}_{\mathbf{E}}\end{pmatrix}\qquad\boldsymbol{\mu}^{\boldsymbol{\delta}_{\mathbf{E}}}=\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{Y}}\\ \boldsymbol{\mu}_{\mathbf{E}}+\boldsymbol{\delta}_{\mathbf{E}}\end{pmatrix}$$

-The variances-covariances between the variables of interest, being the perturbed model $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}^{\boldsymbol{\Delta}\mathbf{YY}})$, or about the variances-covariances between the evidential variables, where the perturbed model is given by $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}^{\boldsymbol{\Delta}\mathbf{EE}})$, or about the covariances between the variables of interest and the evidential variables, being the perturbed model $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}^{\boldsymbol{\Delta}\mathbf{YE}})$, where

$$\boldsymbol{\Sigma}^{\boldsymbol{\Delta}_{\mathbf{YY}}}=\begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{YY}}+\boldsymbol{\Delta}_{\mathbf{YY}} & \boldsymbol{\Sigma}_{\mathbf{YE}}\\ \boldsymbol{\Sigma}_{\mathbf{EY}} & \boldsymbol{\Sigma}_{\mathbf{EE}}\end{pmatrix}\quad\boldsymbol{\Sigma}^{\boldsymbol{\Delta}_{\mathbf{EE}}}=\begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{YY}} & \boldsymbol{\Sigma}_{\mathbf{YE}}\\ \boldsymbol{\Sigma}_{\mathbf{EY}} & \boldsymbol{\Sigma}_{\mathbf{EE}}+\boldsymbol{\Delta}_{\mathbf{EE}}\end{pmatrix}\quad\boldsymbol{\Sigma}^{\boldsymbol{\Delta}_{\mathbf{YE}}}=\begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{YY}} & \boldsymbol{\Sigma}_{\mathbf{YE}}+\boldsymbol{\Delta}_{\mathbf{YE}}\\ \boldsymbol{\Sigma}_{\mathbf{EY}}+\boldsymbol{\Delta}_{\mathbf{EY}} & \boldsymbol{\Sigma}_{\mathbf{EE}}\end{pmatrix}$$

RESULTS

Prop 1. Let $(\mathcal{G},\mathcal{P})$ be a Gaussian Bayesian network with $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\mathbf{X}=\{\mathbf{Y},\mathbf{E}\}$, being $\mathbf{Y}$ the set of variables of interest and $\mathbf{E}$ the set of evidential variables. Giving values to the perturbations of the mean vector $\boldsymbol{\delta}$, the following results are obtained

(1)When the perturbation $\boldsymbol{\delta}_{\mathbf{Y}}$ is added to the mean of the variables of interest the *KL* divergence is given by

$$KL^{\boldsymbol{\mu}_{\mathbf{Y}}}(f,f^{\boldsymbol{\mu}_{\mathbf{Y}}})=\frac{1}{2}\left[\boldsymbol{\delta}_{\mathbf{Y}}^T\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right)^{-1}\boldsymbol{\delta}_{\mathbf{Y}}\right]$$

(2) When the perturbation $\boldsymbol{\delta}_{\mathbf{E}}$ is added to the mean of the variables of interest the *KL* divergence is given by

$$KL^{\boldsymbol{\mu}_{\mathbf{E}}}(f,f^{\boldsymbol{\mu}_{\mathbf{E}}})=\frac{1}{2}\left[\boldsymbol{\delta}_{\mathbf{E}}^T\left(\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)^T\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right)^{-1}\left(\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)\boldsymbol{\delta}_{\mathbf{E}}\right]$$

Prop 2. Let $(\mathcal{G},\mathcal{P})$ be a Gaussian Bayesian network with $\mathbf{X}\sim N(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\mathbf{X}=\{\mathbf{Y},\mathbf{E}\}$, being $\mathbf{Y}$ the set of variables of interest and $\mathbf{E}$ the set of evidential variables. Giving values to the perturbations of the covariance matrix $\boldsymbol{\Delta}=\begin{pmatrix}\boldsymbol{\Delta}_{\mathbf{YY}} & \boldsymbol{\Delta}_{\mathbf{YE}}\\ \boldsymbol{\Delta}_{\mathbf{EY}} & \boldsymbol{\Delta}_{\mathbf{EE}}\end{pmatrix}$, the following results are obtained

(1) When the perturbation $\boldsymbol{\Delta}_{\mathbf{YY}}$ is added to the variances-covariances between variables in $\mathbf{Y}$, the *KL* divergence is

$$KL^{\boldsymbol{\Sigma}_{\mathbf{YY}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YY}}})=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}+\boldsymbol{\Delta}_{\mathbf{YY}}\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}+\boldsymbol{\Delta}_{\mathbf{YY}}\right)^{-1}\right)-\dim(\mathbf{Y})\right]$$

(2) When the perturbation $\boldsymbol{\Delta}_{\mathbf{EE}}$ is added to the variances-covariances between variables in $\mathbf{E}$, the *KL* divergence is

$$KL^{\boldsymbol{\Sigma}_{\mathbf{EE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{EE}}})=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E},\boldsymbol{\Delta}_{\mathbf{EE}}}\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E},\boldsymbol{\Delta}_{\mathbf{EE}}}\right)^{-1}\right)+\left(\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E},\boldsymbol{\Delta}_{\mathbf{EE}}}-\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E}}\right)^T\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E},\boldsymbol{\Delta}_{\mathbf{EE}}}\right)^{-1}\left(\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E},\boldsymbol{\Delta}_{\mathbf{EE}}}-\boldsymbol{\mu}^{\mathbf{Y}|\mathbf{E}}\right)-\dim(\mathbf{Y})\right]$$

(3) When the perturbation $\boldsymbol{\Delta}_{\mathbf{YE}}$ is added to the covariances between variables in $\mathbf{Y}$ and variables in $\mathbf{E}$, the obtained *KL* divergence is given by

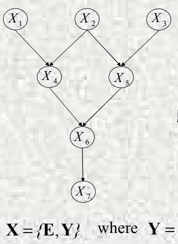
$$KL^{\boldsymbol{\Sigma}_{\mathbf{YE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YE}}})=\frac{1}{2}\left[\ln\frac{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}-M(\boldsymbol{\Delta}_{\mathbf{YE}})\right|}{\left|\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\right|}+tr\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}-M(\boldsymbol{\Delta}_{\mathbf{YE}})\right)^{-1}\right)+(\mathbf{e}-\boldsymbol{\mu}_{\mathbf{E}})^T\left(\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)^T\boldsymbol{\Delta}_{\mathbf{YE}}^T\left(\boldsymbol{\Sigma}^{\mathbf{Y}|\mathbf{E}}-M(\boldsymbol{\Delta}_{\mathbf{YE}})\right)^{-1}\boldsymbol{\Delta}_{\mathbf{YE}}\left(\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\right)(\mathbf{e}-\boldsymbol{\mu}_{\mathbf{E}})-\dim(\mathbf{Y})\right]$$

where $M(\boldsymbol{\Delta}_{\mathbf{YE}})=\boldsymbol{\Delta}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Sigma}_{\mathbf{YE}}^T+\boldsymbol{\Sigma}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Delta}_{\mathbf{EY}}+\boldsymbol{\Delta}_{\mathbf{YE}}\boldsymbol{\Sigma}_{\mathbf{EE}}^{-1}\boldsymbol{\Delta}_{\mathbf{EY}}$

EXAMPLE

The Gaussian Bayesian network of the example shows the duration of the different components of a 7-elements machine

$$\mathbf{X}=\{X_1,X_2,X_3,X_4,X_5,X_6,X_7\}\approx N(\mathbf{x}\mid\mu,\Sigma)$$



$$\boldsymbol{\mu}=\begin{pmatrix}1\\3\\2\\1\\4\\5\\8\end{pmatrix}\qquad\Sigma=\begin{pmatrix}1&0&0&1&0&2&2\\0&1&0&2&2&8&8\\0&0&2&0&2&4&4\\1&2&0&6&4&20&20\\0&2&2&4&10&28&28\\2&8&4&20&28&97&97\\2&8&4&20&28&97&99\end{pmatrix}$$

$\mathbf{X}=\{\mathbf{E},\mathbf{Y}\}$ where $\mathbf{Y}=\{X_4,X_5,X_6,X_7\}$ $\quad\mathbf{E}=\{X_1=2,X_2=2,X_3=1\}$

After quantifying the uncertainty of the parameters, it could be concluded the mean vector perturbations $\boldsymbol{\delta}$ and the covariance matrix perturbations $\boldsymbol{\Delta}$ are given by

$$\boldsymbol{\delta}=\begin{pmatrix}0\\-1\\1\\0\\1\\0\\-1\end{pmatrix}\qquad\boldsymbol{\Delta}=\begin{pmatrix}0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&-1&0&0&0&-1\\0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&0&0&0&1&-2\\0&0&-1&1&0&-2&0\end{pmatrix}$$

Then we obtain that *the network is sensitive to the perturbations proposed for the mean vector* μ *and for* $\boldsymbol{\Sigma}_{\mathbf{YY}}$ by computing the values of the *KL* divergence from the expressions in **Prop 1** and **Prop 2**

$$KL^{\boldsymbol{\mu}_{\mathbf{Y}}}(f,f^{\boldsymbol{\mu}_{\mathbf{Y}}})=2.375$$
$$KL^{\boldsymbol{\mu}_{\mathbf{E}}}(f,f^{\boldsymbol{\mu}_{\mathbf{E}}})=2.125$$
$$KL^{\boldsymbol{\Sigma}_{\mathbf{YY}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YY}}})=1.629$$
$$KL^{\boldsymbol{\Sigma}_{\mathbf{EE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{EE}}})=0.596$$
$$KL^{\boldsymbol{\Sigma}_{\mathbf{YE}}}(f,f^{\boldsymbol{\Sigma}_{\mathbf{YE}}})=0.265$$