Marginals of DAG-Isomorphic Independence Models

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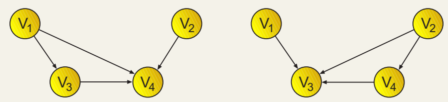
Marginals of Independence Models

Question: If a (probabilistic) independence model is not a DAG-isomorph, is it possible to turn it into a DAG-isomorph by introducing latent variables?

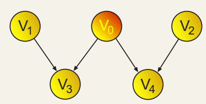
Example

I defined by

- (S1): $I(V_1,V_2 \mid \emptyset)$
- (S2): $I(V_1,V_2 \mid V_3)$
- (S3): $I(V_2,V_3 \mid \emptyset)$
- (S4): $I(V_1,V_4 \mid \emptyset)$
- (S5): $I(V_1,V_2 \mid V_4)$
- (S6): $I(V_2,V_3 \mid V_1)$
- (S7): $I(V_1,V_4 \mid V_2)$



Two minimal I-maps for I



A latent P-map for I

Marginal of an independence model

- Let I be an independence model on V and $A \subseteq V$.
- The marginal of I on A is $\{ I(X,Y|Z) \in I \mid X, Y, Z \subseteq A \}$

Definition: DAG-isomorph marginal and latent perfect map

- Let I be an independence model on a set of variables V
- If there exists
  - A set $V^* \supseteq V$
  - A DAG-isomorph independence model I* on V*
  - which has graph G* as perfect map
  - and I as marginal independence model,
- then I is called a DAG-isomorph marginal and G* a latent P-map.

Independence Models

1. Probabilistic conditional independence

- Based on classical notion of probabilistic independence
- $I_P(X,Y|Z)$ if for every $x \in X$, $y \in Y$, $z \in Z$: $P^{XYZ}(x,y,z) * P^Z(z) = P^{XZ}(x,z) * P^{YZ}(y,z)$

2. Graphical independence models (DAG)

- Based on concept of d-separation
- $I_G(X,Y|Z)$ if Z d-separates X and Y in graph

3. Semi-graphoid independence models

- Triplets that satisfy semi-graphoid axioms:
  - $I(X,Y|Z) \Rightarrow I(Y,X|Z)$
  - $I(X,YW|Z) \Rightarrow I(X,Y|Z)$
  - $I(X,YW|Z) \Rightarrow I(X,Y|ZW)$
  - $I(X,Y|Z) \wedge I(X,W|YZ) \Rightarrow I(X,YW|Z)$

Probabilistic ↔ Graphical independence

- I-maps (Independence map):
  - $I_G(X,Y|Z) \Rightarrow I_P(X,Y|Z)$
- P-maps (Perfect map):
  - $I_G(X,Y|Z) \Leftrightarrow I_P(X,Y|Z)$
- DAG-isomorphism
  - An independence model I is DAG-isomorph if there exists a graph G=(V,A) that is a perfect map for I.
- Necessary conditions for DAG-isomorph:
  - $I(X,Y|Z) \wedge I(X,Y|Z\gamma) \Rightarrow I(X,\gamma|Z) \vee I(Y,\gamma|Z)$
  - $I(\alpha,\beta|\gamma\delta) \wedge I(\gamma,\delta|\alpha\beta) \Rightarrow I(\alpha,\beta|\gamma) \vee I(\alpha,\beta|\delta)$

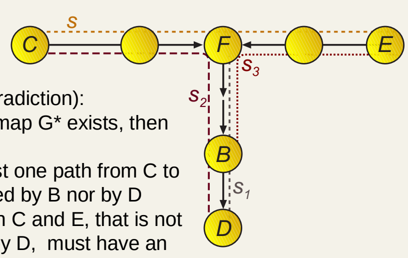
Proof

Let *I* be the following independence model on V = {B,C,D,E}:

- (T1) I(B,E | CD)
- (T2) $I(C,E \mid \emptyset)$
- (T3) I(C,D | B)

then there exists no latent P-map for I.

- I satisfies the necessary conditions for DAG-isomorphism of Pearl.
- There exists a probability distribution on V that has I as its probabilistic independence model



Sketch of proof (by contradiction):
Assume that a latent P-map G* exists, then

- There exists at least one path from C to E, that is not blocked by B nor by D
- Every path between C and E, that is not blocked by B, nor by D, must have an infinite number of (converging) nodes.

Simpler proof is possible for smaller class of independence models: Construct an independence model that is not weakly transitive.

Main Results and Further Research

Main results

- There exists an independence model that
  - satisfies the necessary conditions for DAG-isomorphism, but
  - is not a DAG-isomorph marginal
- There exists a probability distribution that has exactly this independence model.

Future research

- Necessary and sufficient conditions for DAG-isomorph marginals

References

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